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Infrared Correlations in de Sitter Space: Field Theoretic vs. Stochastic Approach

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Abstract

We consider massive $\lambda\phi^4$ theory in de Sitter background. The mass of the scalar field ϕ is chosen small enough, such that the amplification of superhorizon momentum modes leads to a significant enhancement of infrared correlations, but large enough such that perturbation theory remains valid. Using the Closed-Time-Path approach, we calculate the infrared corrections to the two-point function of ϕ to 2-loop order. To this approximation, we find agreement with the correlation found using stochastic methods. When breaking the results down to individual Feynman diagrams obtained by the two different methods, we observe that these agree as well.

1 Introduction

For a free scalar field, that couples to gravitation minimally, there exists no de Sitter invariant vacuum state for which the propagator exhibits the light-cone singularities that are required for a physical massless field [1, 2]. This is because of the amplification of momentum modes that exit the de Sitter horizon. Due to the redshift, soft modes accumulate on superhorizon scales formally resulting in an infrared (IR) divergence of the propagator, see *e.g.* Ref. [3]. While it is not clear whether massless scalar particles are realized in Nature, they can serve as a toy model for gravitons, the propagator of which exhibits similar IR divergences [3].

The absence of a de Sitter invariant vacuum for a massless, minimally coupled, free scalar field is however physically irrelevant, as there are no interactions which can be used to probe its quantum state. A more interesting and challenging question is whether there is a de Sitter invariant quantum state for an *interacting* scalar field, and it is this question that has drawn the interest of a number of authors who have addressed the problem using a wide range of methods [4–35].

The model that has been most widely studied in this respect is ϕ^4 theory, which is specified by the Lagrangian

$$\mathcal{L} = \sqrt{-g} \left[\frac{1}{2} g^{\mu\nu} (\partial_\mu \phi) (\partial_\nu \phi) - V(\phi) \right], \quad (1)$$

where the potential is

$$V(\phi) = \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4. \quad (2)$$

The field ϕ couples to the de Sitter background through the metric tensor $g^{\mu\nu}$. De Sitter space is parametrised by the Hubble expansion rate H , and various possible choices for the coordinates are presented in Ref [36]. For the present work, we find it useful to use conformal coordinates on the expanding patch of de Sitter space that are given in Eq. (21) below.

While $m = 0$ in the aforementioned massless ϕ^4 theory, we will take here a non-vanishing mass. The model we consider thus relies on three parameters: H , λ and m . The reason for introducing the mass m is that this parameter can force the perturbative loop expansion to be valid. In Minkowski space, perturbation expansion can be performed provided $|\lambda| < 4\pi$. However, in de Sitter space and when $m \ll H$, this is no longer true due to the enhancement of IR modes of the scalar field ϕ . As we discuss below, the parametric region where there is a significant IR enhancement of superhorizon modes but, at the same time, perturbation theory remains valid, is given by

$$m^2 \ll H^2 \quad \text{and} \quad \lambda \ll m^4/H^4. \quad (3)$$

We refer to the model in this parametric domain as light, perturbative ϕ^4 theory in de Sitter space.

The quantity that we aim to calculate is the fluctuation of ϕ , *i.e.* the expectation value $\langle \phi^2 \rangle$. We suppress here the space-time coordinates of the field operators, but imply that the separation of the two coordinates should be taken to be of superhorizon scale, $\gtrsim H^{-1}$, what we specify more precisely in the calculations below. In Section 3, we pursue the direct approach to this problem, which is to use Quantum Field Theory (QFT) and to calculate the loop corrections to the the propagator. The background of a curved space-time suggests to use the Closed-Time-Path (CTP) formalism for this purpose. Such a calculation must in particular address two points:

- In the massless ϕ^4 model, the loop expansion appears not to converge, as can be seen from the corresponding problem in Euclidean de Sitter space and as it is also indicated for Lorentzian de Sitter space in the present work. Promoting the scalar field to an $O(N)$ multiplet, a $1/N$ expansion can be performed and be truncated at the zeroth order, which includes only the one-loop seagull diagram in Figure 1(A). This calculation has been performed in Ref. [8] and confirmed in some subsequent studies [15, 17, 20]. To our knowledge, an extension to order $1/N$

or beyond has not yet been performed. As stated above, in the present work we force the convergence of the loop expansion through a non-vanishing mass term that satisfies the condition (3).

- The leading order correction in both the $O(N)$ symmetric model as well as in the light, perturbative ϕ^4 model is given by the seagull diagram [Figure 1(A)]. It plays a special role, because it is a local correction and therefore can be absorbed in the redefinition of the local mass term. At higher orders, there occur non-local diagrams as well, and there is no agreement in the current literature about how to correctly evaluate these. In the light massive model we need to evaluate the sunset diagram [Figure 1(D)] for a consistent calculation of the fluctuation of ϕ to $\mathcal{O}(\lambda^2)$. Here, we do so by evaluating, in a rather straightforward manner, the convolution integrals corresponding to the sunset diagram, which appear in the Schwinger-Dyson equation. For this procedure, it turns out to be crucial to account for the decay of the IR fluctuations of the scalar field at very large distances. The computation of the $\mathcal{O}(\lambda^2)$ correction to the fluctuation of ϕ is therefore the main technical result presented here.

It is desirable to perform a consistency check of the QFT calculation. Substantial progress has been made for massless ϕ^4 theory in Euclidean de Sitter space, where an invariant quantum state is derived to leading IR order in Ref. [14]. This calculation is confirmed in Ref. [27], where a loop expansion and the necessary resummation are performed. Moreover, it is pointed out there that the Euclidean QFT result agrees with what is obtained from the stochastic approach for scalar field fluctuations in Lorentzian de Sitter space [37, 38]. Besides, there is work arguing that in de Sitter space Euclidean two-point functions can be analytically continued in order to obtain their Lorentzian counterparts [39]. Therefore, it appears interesting to compare the QFT result with the stochastic result. For this purpose, we formulate in Section 2 the stochastic approach in terms of a diagrammatic expansion that bears a close relation to the CTP diagrams and indeed find agreement to order $\mathcal{O}(\lambda^2)$. In fact, the agreement extends to a diagram-by-diagram comparison between the two approaches.

We emphasise here that both the QFT calculation (using the CTP formalism) and the stochastic approach are based on the field quantisation of ϕ . The primary purpose of the nomenclature “QFT” and “stochastic” is therefore to distinguish between these two methods. However, it also reflects the fact that in the QFT approach no assumption about the classical behaviour of the IR modes is made, in contrast to the stochastic method.

A setup similar to the one specified by the Lagrangian (1) and the relations (3) is studied within Ref. [31]. Although the methodology agrees to some extent with what is used here in that the CTP formalism is employed, there are differences in the details of the calculation and in the choice of the quantities that are presented as the final results. A perturbative calculation in massless ϕ^4 theory is also valid at early times, when using a de Sitter breaking propagator with IR correlations, that grow in time, see *e.g.* Refs. [4–7]. In this setup, the agreement between stochastic and QFT results at order λ has been

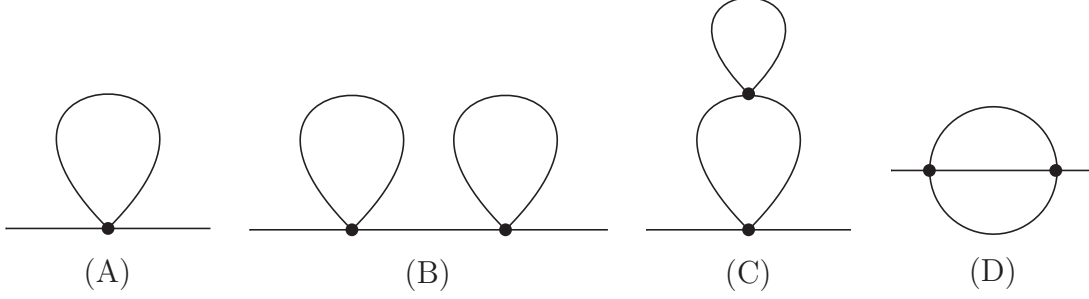


Figure 1: The diagrammatic contributions to the self energy up to order λ^2 . When amputating the external lines, Diagram (A) corresponds to the seagull-type self-energy $i\Pi_{\text{sg}}$, Diagram (C) to the cactus-type self-energy $i\Pi_{\text{ca}}$ and Diagram (D) to the sunset-type self-energy $i\Pi_{\text{ss}}$. Accordingly, for the diagrammatic decomposition within the stochastic approach, we denote Diagram (A) by $\langle\phi^2\rangle_{\text{sg}}$, Diagram (B) by $\langle\phi^2\rangle_{\text{sgsg}}$, Diagram (C) by $\langle\phi^2\rangle_{\text{ca}}$ and Diagram (D) by $\langle\phi^2\rangle_{\text{ss}}$.

noted in Ref. [7]. For other earlier discussions of the stochastic-QFT correspondence see e.g [9, 40].

For most of the present discussion, we work in four space-time dimensions. This way, we avoid the recurring notation of factors, that account for a general dimensionality. The generalisation to D space-time dimensions is however straightforward, and a brief discussion along with the main results is presented in Section 4.

2 Stochastic Approach

In the stochastic approach [37], the field ϕ is separated into a long wavelength (*i.e.* superhorizon) part, that is treated as a classical stochastic variable, and a short wavelength part for which the underlying description as a quantum field is maintained. For simplicity *we here denote the long wavelength part by ϕ when we refer to the stochastic approach*, using the same symbol as for the underlying field. When it is assumed that the behaviour of the long wavelength modes is classical, their dynamics is driven by a stochastic noise induced by the quantum short-wavelength modes. In particular the stochastic theory of inflationary dynamics is based on the Starobinsky equation

$$\dot{\phi} + \frac{\partial_{\phi} V}{3H} = \xi(t), \quad (4)$$

a Langevin-type equation for the scalar field where ξ is a Gaussian random force with

$$\langle\xi(t)\xi(t')\rangle = \frac{H^3}{4\pi^2}\delta(t-t'). \quad (5)$$

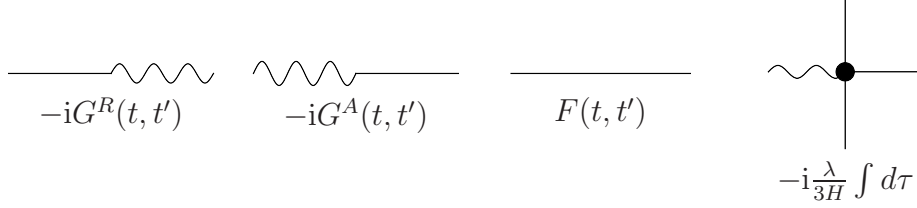


Figure 2: The elements out of which stochastic diagrams are constructed. The choice of vertex factor implies that the assembled diagrams should be divided by their symmetry factor.

The expectation value of an operator $\mathcal{O}[\phi]$ is given by¹

$$\langle \mathcal{O}[\phi] \rangle = \int D[\xi] e^{-\frac{1}{2} \int dt \xi^2 \frac{4\pi^2}{H^3}} \int D[\phi] \mathcal{O}[\phi] \delta\left(\dot{\phi} + \partial_\phi V/3H - \xi\right). \quad (6)$$

By expressing the delta functional as a functional “Fourier transform” with the aid of an auxiliary field ψ and performing the Gaussian ξ integral we obtain

$$\langle \mathcal{O}[\phi] \rangle = \int D[\phi, \psi] \mathcal{O}[\phi] e^{-\int dt \left[i\psi \left(\dot{\phi} + \frac{\partial_\phi V}{3H} \right) + \frac{H^3}{8\pi^2} \psi^2 \right]}. \quad (7)$$

Let us now focus on the quadratic potential $V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4$. To obtain a diagrammatic expansion we rewrite the action by bringing the quadratic part in a more symmetric form²

$$\langle \mathcal{O}[\phi] \rangle = \int D[\phi, \psi] \mathcal{O}[\phi] e^{-i \int dt \left[\frac{1}{2} (\phi, \psi) \begin{pmatrix} 0 & -\partial_t + \frac{m^2}{3H} \\ \partial_t + \frac{m^2}{3H} & -i \frac{H^3}{4\pi^2} \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} + \frac{\lambda}{3!} \frac{\psi \phi^3}{3H} \right]}. \quad (8)$$

The free correlation functions are then determined as the functional and matrix inverse of the quadratic operator:

$$\begin{pmatrix} \langle \phi(t)\phi(t') \rangle & \langle \phi(t)\psi(t') \rangle \\ \langle \psi(t)\phi(t') \rangle & \langle \psi(t)\psi(t') \rangle \end{pmatrix} \equiv -i \begin{pmatrix} 0 & \left(-\partial_t + \frac{m^2}{3H} \right) \\ \left(\partial_t + \frac{m^2}{3H} \right) & -i \frac{H^3}{4\pi^2} \end{pmatrix}^{-1} \delta(t - t') = \begin{pmatrix} F(t, t') & -iG^R(t, t') \\ -iG^A(t, t') & 0 \end{pmatrix}. \quad (9)$$

Here $G^{(R,A)}(t, t')$ are the retarded and advanced Green functions for the operator $\partial_t + \frac{m^2}{3H}$

$$G^R(t, t') = G^A(t', t) = e^{-\frac{m^2}{3H}(t-t')} \Theta(t - t'), \quad (10)$$

¹Note that we normalise the functional integration measures that appear in Eq. (6) such that $\langle \mathbb{1} \rangle = 1$. This also corresponds to the retarded Ito regularisation of the stochastic equation (4) [41].

²In fact, the analogue of Eq. (8) containing second time derivatives can be obtained directly from the more fundamental CTP path integral after short wavelength modes are integrated out [42, 43].

and $F(t, t') \equiv \langle \phi(t) \phi(t') \rangle$ is the 2-point function of ϕ

$$F(t, t') = \frac{H^3}{4\pi^2} \int_0^{+\infty} d\tau G^R(t, \tau) G^A(\tau, t') = \frac{3H^4}{8\pi^2 m^2} \left(e^{-\frac{m^2}{3H}|t-t'|} - e^{-\frac{m^2}{3H}(t+t')} \right). \quad (11)$$

If t and t' are taken to be sufficiently large, or, equivalently, the stochastic process is taken to have begun early enough, the correlator reduces to

$$F(t, t') \simeq \frac{3H^4}{8\pi^2 m^2} e^{-\frac{m^2}{3H}|t-t'|} \quad (12)$$

which is the form that we'll be using from now on. Note that in the massless limit, $G^R(t, t') \rightarrow \Theta(t - t')$, and the variance, as is well known [44], grows linearly with time

$$\langle \phi^2(t) \rangle_{m=0} \simeq \frac{H^3}{4\pi^2} t. \quad (13)$$

We can now construct a diagrammatic expansion out of the elements shown in Figure 2: there are three types of propagator, $-iG^R(t, t')$, $-iG^A(t, t')$ and $F(t, t')$ along with the relation $G^R(t, t') = G^A(t', t)$ and a single vertex with one wiggly and three solid legs. A vertex refers to an internal time variable which is integrated over. Note that the diagrammatic elements are identical in form to those of the CTP formalism, expressed in the Keldysh basis, but with the additional three-wiggle-line vertex omitted. The absence of the latter vertex corresponds to the semiclassical nature of the result. In this work we focus on the two-point function to second order, given by the diagrams of Figure 3. Note that vacuum bubble diagrams are zero. To facilitate later comparison with the QFT calculation, we break down the result into the individual Feynman diagrams depicted in Figure 3. Setting $t = t'$ and taking the late time limit we obtain:

$$\langle \phi^2 \rangle_{\text{sg}} = -\lambda \frac{9H^8}{128\pi^4 m^6}, \quad (14a)$$

$$\langle \phi^2 \rangle_{\text{ca}} = \lambda^2 \frac{27H^{12}}{2048\pi^6 m^{10}}, \quad (14b)$$

$$\langle \phi^2 \rangle_{\text{sgsg}} = \lambda^2 \frac{27H^{12}}{2048\pi^6 m^{10}}, \quad (14c)$$

$$\langle \phi^2 \rangle_{\text{ss}} = \lambda^2 \frac{9H^{12}}{1024\pi^6 m^{10}}. \quad (14d)$$

Adding up the individual contributions, we obtain

$$\lim_{t \rightarrow \infty} \langle \phi(t)^2 \rangle = \frac{3H^4}{8\pi^2 m^2} - \lambda \frac{9H^8}{128\pi^4 m^6} + \lambda^2 \frac{9H^{12}}{256\pi^6 m^{10}}. \quad (15)$$

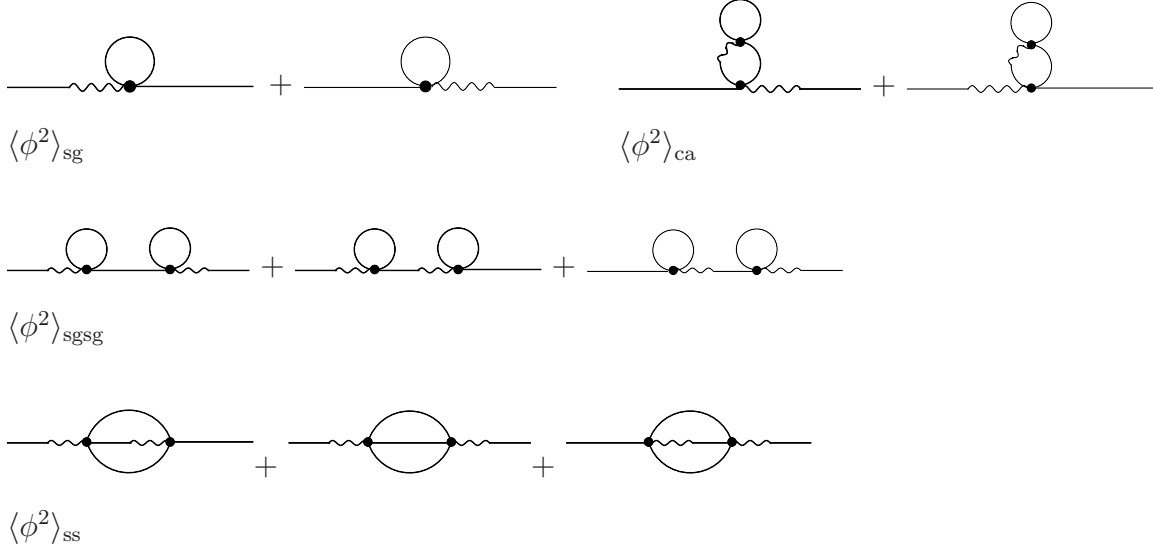


Figure 3: The stochastic diagrams contributing to $\langle \phi(t)\phi(t') \rangle$ up to order λ^2 . Vacuum bubbles are zero by construction. Note that they are identical in form to the CTP diagrams in the Keldysh basis but with the three-wiggle vertex removed. This corresponds to a semiclassical approximation. The labels refer to the topology of the diagrams: seagull, cactus, double seagull and sunset.

As we will see below, the result (15) agrees with the result from the QFT Schwinger-Dyson equations. Furthermore, each individual contribution from the topologically distinct diagrams, Eqs. (14a)-(14d), equals the QFT contribution from diagrams of corresponding topology.

We should note here a different way in which the result (15) can be obtained. At late times the stochastic process (4) is described by the probability distribution function [37, 38]

$$\varrho(\phi) = \mathcal{N} e^{-\frac{8\pi^2}{3H^4} V(\phi)}, \quad (16)$$

where the normalisation \mathcal{N} is determined by the condition

$$\int_{-\infty}^{\infty} d\phi \varrho(\phi) = 1. \quad (17)$$

Expectation values (at equal times) are obtained using the probability distribution function in the usual way, for example

$$\langle \phi^n \rangle = \int_{-\infty}^{\infty} d\phi \phi^n \varrho(\phi). \quad (18)$$

For the particular potential (2), we can expand

$$e^{-\frac{8\pi^2}{3H^4}V(\phi)} = e^{-\frac{4\pi^2}{3H^4}m^2\phi^2} \left(1 - \frac{\pi^2\lambda\phi^4}{9H^4} + \frac{\pi^4\lambda^2\phi^8}{162H^8} + \dots \right). \quad (19)$$

Substituting this into Eq. (18), we perform the integrals for the fluctuation and the normalisation with the result

$$\langle\phi^2\rangle = \frac{3H^4}{8\pi^2m^2} - \frac{9\lambda H^8}{128\pi^4m^6} + \frac{9\lambda^2 H^{12}}{256\pi^6m^{10}} + \dots, \quad (20)$$

which coincides with Eq. (15). Again, the expansion (20) has an immediate interpretation in terms of Feynman diagrams: For each vertex, we assign a factor $-\lambda 8\pi^2/(3H^4)$ and for each propagator, a factor $3H^4/(8\pi^2m^2)$. Moreover, we divide by the appropriate symmetry factor. We thus obtain for the seagull diagram [Figure 1(A), symmetry factor 2] $\langle\phi^2\rangle_{\text{sg}}$, for the diagram in Figure 1(B) (symmetry factor 4) $\langle\phi^2\rangle_{\text{sgsg}}$, for the cactus diagram [Figure 1(C), symmetry factor 4] $\langle\phi^2\rangle_{\text{ca}}$, and for the sunset diagram [Figure 1(C), symmetry factor 6] $\langle\phi^2\rangle_{\text{ss}}$, where all results can be found in Eqs. (14).

Apparently, the expansion (15) is valid provided $\lambda \ll m^4/H^4$, in accordance with relation (3). However, the stochastic theory implies that the regime with $\lambda > m^4/H^4$ is also meaningful since the series can be summed and correlation functions for the potential (2) can be evaluated exactly using Eq (16). In this case the integrals leading to $\langle\phi^2\rangle$ can be evaluated in terms of Bessel and of Hypergeometric functions.

Before closing this Section on the stochastic approach to inflation, we discuss the spatial correlations of the stochastic field. We use conformal coordinates with the metric tensor

$$g_{\mu\nu}(x) = a^2(\eta)\text{diag}(1, -1, -1, -1). \quad (21)$$

The scale factor is given by $a(\eta) = -1/(H\eta)$, and $\eta = x^0$ is the conformal time. Below, we sometimes write $a(x) \equiv a(x^0)$ for a 4-vector x . For the expanding de Sitter space, $\eta \in (-\infty; 0)$. Now we first consider two points \mathbf{x} and \mathbf{x}' that begin at a time η_0 with the same field value and that are initially separated by a physical distance $\Delta r(\eta_0) = a(\eta_0)|\Delta\mathbf{x}| \sim 1/H$, which is the smallest distance for which the stochastic description is meaningful and where $\Delta\mathbf{x} = \mathbf{x} - \mathbf{x}'$. The subsequent evolution of the field ϕ will be given by Eq. (4) for two different realisations of the noise field ξ . From Eq. (11) we see that the field values at these two points will be completely uncorrelated after a physical time interval of $\Delta t \sim \frac{H}{m^2}$, where the physical time t is related to the conformal time through $dt = a(\eta)d\eta$. Here, we fix the physical time coordinate by setting $a(t) \equiv a(\eta) = a_0 e^{Ht}$, and accordingly for the time variables with primes or subscripts. The physical separation of these two points will then be $\Delta r(\eta) \sim \frac{1}{H} e^{\frac{H^2}{m^2}t}$, where $t = t_0 + \Delta t$. Thus the field ϕ maintains its coherence over distances that satisfy

$$1 < a^2(\eta)H^2\Delta\mathbf{x}^2 \ll e^{\frac{2H^2}{m^2}t}, \quad (22)$$

while it is incoherent on physical scales $\Delta r > \frac{1}{H}e^{\frac{H^2}{m^2}}$. Beyond this general statement, it is in fact possible to obtain the dependence of the correlation function on spatial separation by invoking de Sitter invariance. When we write $\Delta x = x - x'$, such that

$$\Delta x^2 = (\eta - \eta')^2 - (\mathbf{x} - \mathbf{x}')^2, \quad (23)$$

where $\mathbf{x} = (x^1, x^2, x^3)$, we can define a de Sitter invariant length function as [4, 45]

$$y(x; x') = a(\eta)a(\eta')H^2\Delta x^2 = \frac{\Delta x^2}{\eta\eta'} = -4\sin^2\left(\frac{1}{2}H\ell(x; x')\right). \quad (24)$$

We have indicated here the relation to the length $\ell(x; x')$ along a geodesic that connects these points. Note that for $y > 0$, $\ell(x; x')$ is purely imaginary, corresponding to time-like separations. Space-like separations correspond to $-4 < y < 0$, where ℓ is real, while for $y < -4$, there is no geodesic that connects the two points (even though a complex ℓ may still be defined), see *e.g.* Ref. [45]. Due to de Sitter invariance, the correlations of ϕ should be functions of y only. For large physical time separations $t - t' \gg 1/H$, we can approximate

$$y(x; x') \approx e^{H(t-t')} - e^{H(t+t')}a_0^2H^2|\Delta\mathbf{x}|^2, \quad (25)$$

while for separations with $t = t'$

$$y(x; x') = -e^{2Ht}a_0^2H^2|\Delta\mathbf{x}|^2. \quad (26)$$

We therefore replace $e^{-H(t-t')} \rightarrow \frac{1}{a^2(t)H^2|\Delta\mathbf{x}|^2}$ in Eq. (11) to obtain

$$\langle\phi(t, \mathbf{x})\phi(t, \mathbf{x}')\rangle = \frac{3H^4}{8\pi^2m^2}\left(\frac{1}{a^2(t)H^2|\mathbf{x} - \mathbf{x}'|^2}\right)^{\frac{m^2}{3H^2}}. \quad (27)$$

We thus see a mild power-law decay at large distances. This relation is verified in the the following Section, see Eq. (31). Besides, we have also ignored here the sign of y , which will be properly accounted for in the QFT approach as well.

3 Field Theory Approach

3.1 Propagator for a Massive Free Field

We now pursue the QFT approach to light, perturbative ϕ^4 theory in de Sitter space. Due to the time-dependent background, it is pertinent to use the CTP approach. As by the relations (3), the problem is perturbative and we can pursue a loop expansion that we truncate here at the two-loop order. The result will be a perturbatively improved propagator that we can compare with the fluctuation of ϕ that is obtained by stochastic means, Eq. (15).

The basic building block of the Feynman diagrams is the free propagator $i\Delta^{(0)}$, that satisfies the Klein-Gordon equation

$$a^4 (\nabla_x^2 - m^2) i\Delta^{(0)fg}(x; x') = fg\delta^{fg} i\delta^4(x - x'), \quad (28)$$

where $f, g = \pm$ are CTP indices and $(\nabla_x)_\mu$ is the covariant derivative with respect to x . Note that the IR effects in de Sitter space can at least partly be accounted for by a dynamical mass m_{dyn} [15, 27, 31]. When this is the case, the leading IR effects are captured by a full propagator $i\Delta$ that also satisfies the Klein-Gordon equation with m replaced by m_{dyn} . Therefore, Eq. (28) and its solution also describe important properties of the full propagator in the interacting theory.

The causal properties of two-point functions are accounted for by the following ε -prescriptions:

$$\Delta x^{T^2}(x; x') = \Delta x^{++^2}(x; x') = (|\eta - \eta'| - i\varepsilon)^2 - |\mathbf{x} - \mathbf{x}'|^2, \quad (29a)$$

$$\Delta x^{<^2}(x; x') = \Delta x^{+-^2}(x; x') = (\eta - \eta' + i\varepsilon)^2 - |\mathbf{x} - \mathbf{x}'|^2, \quad (29b)$$

$$\Delta x^{>^2}(x; x') = \Delta x^{-+^2}(x; x') = (\eta - \eta' - i\varepsilon)^2 - |\mathbf{x} - \mathbf{x}'|^2, \quad (29c)$$

$$\Delta x^{\bar{T}^2}(x; x') = \Delta x^{--^2}(x; x') = (|\eta - \eta'| + i\varepsilon)^2 - |\mathbf{x} - \mathbf{x}'|^2. \quad (29d)$$

The superscripts \pm are CTP indices that are directly inherited by the length function through its definition (24). The superscript T (\bar{T}) indicates (anti-) time ordering, whereas for the superscript $>$ ($<$), operators evaluated at the coordinate x (x') appear on the left (right) within the expression for an expectation value. For more details on the CTP formalism and its application to quantum fields in de Sitter space, see Refs. [4, 15, 46].

The length function (24) is useful in order to keep de Sitter invariance manifest whenever that is possible. In particular, we can express the Klein-Gordon equation (28) as

$$a^4(x)H^2 \left[-4y \left(1 + \frac{y}{4} \right) \frac{d^2}{dy^2} - 8 \left(1 + \frac{y}{2} \right) \frac{d}{dy} - \frac{m^2}{H^2} \right] i\Delta^{(0)fg}(y(x; x')) = fg\delta^{fg} i\delta^4(\Delta x), \quad (30)$$

where the exact solution is given by Eq. (A.1). Throughout this work, we are interested the situation where $m \ll H$, such that sizeable IR fluctuations in the field ϕ occur due to the expansion of the Universe. In this limit, we can use the approximation

$$i\Delta^{(0)fg}(y) = \frac{H^2}{4\pi^2} \left[-\frac{1}{y^{fg}} + \frac{3H^2}{2m^2} \left(-\frac{1}{y^{fg}} \right)^{\frac{1}{3} \frac{m^2}{H^2}} + \mathcal{O} \left(y^{-2} \frac{m^2}{H^2} \right) \right], \quad (31)$$

which follows from the expansion (A.5). (The higher order terms are suppressed by powers of m^2/H^2 and, at large distances, by additional powers of $1/y$.) The second term in the square brackets is IR enhanced due to the relation (3). In order to calculate the

leading IR enhanced corrections to the field fluctuation, we need to collect the contribution from the highest power of the IR enhanced factor at each order in λ . As we show below, additional IR enhanced factors result from the space-time integration.

The ε -prescriptions (29) determine how to evaluate this solution. In particular, for time-like ($y > 0$) separations, the propagator $i\Delta^{(0)}$ acquires an imaginary part. This can be most conveniently isolated when making the following approximation (valid for $m^2 \ll H^2$):

$$\frac{3H^2}{2m^2}(-y)^{-\frac{n}{3}\frac{m^2}{H^2}} = \frac{3H^2}{2m^2} \left(1 - i\frac{n}{3}\frac{m^2}{H^2} \arg(-y) \right) |y|^{-\frac{n}{3}\frac{m^2}{H^2}} + \mathcal{O}\left(\frac{m^2}{H^2}\right), \quad (32)$$

where n is an integer number (arising from powers of the propagator that occur in Feynman diagrams) and where we have suppressed the CTP indices that are responsible for the infinitesimal phase of y . It is useful to note that

$$|y|^{-\frac{n}{3}\frac{m^2}{H^2}} = 1 - \frac{n}{3}\frac{m^2}{H^2} \log |y| + \mathcal{O}\left(\left(\frac{n}{3}\frac{m^2}{H^2} \log |y|\right)^2\right). \quad (33)$$

We see that all basic propagators $i\Delta^{(0)++}$, $i\Delta^{(0)-+}$, $i\Delta^{(0)+-}$ and $i\Delta^{(0)--}$ therefore contain an IR-enhanced contribution $3H^4/(8\pi^2 m^2)$ for

$$|y| \ll \exp(3H^2/m^2). \quad (34)$$

For larger values of $|y|$, the IR-enhanced terms decay following a mild power law. We discuss below, that this mild decay is however crucial in order to regulate the integrals that occur within the Schwinger-Dyson equations. Here, we note in addition that this decay at large distances is of the same physical origin as the spectral tilt that the inflationary power spectrum acquires from the η -parameter [47].

While the basic propagators all contain IR-enhanced contributions, within the causal propagators, *i.e.* the retarded and advanced ones

$$i\Delta^{(0)R,A}(x; x') = i\Delta^{(0)T}(x; x') - i\Delta^{(0)<,>}(x; x'), \quad (35)$$

the IR-enhanced terms cancel. To obtain a useful representation for the causal propagators, we note with the help of the approximation (32), that for time-like separations, the propagators receive non-vanishing imaginary parts through

$$i \arg(-y^{++}(x; x')) = i\pi\vartheta(\Delta x^2), \quad (36a)$$

$$i \arg(-y^{-+}(x; x')) = i\pi\vartheta(\Delta x^2)\text{sign}(\Delta x^0), \quad (36b)$$

$$i \arg(-y^{+-}(x; x')) = -i\pi\vartheta(\Delta x^2)\text{sign}(\Delta x^0), \quad (36c)$$

$$i \arg(-y^{--}(x; x')) = -i\pi\vartheta(\Delta x^2), \quad (36d)$$

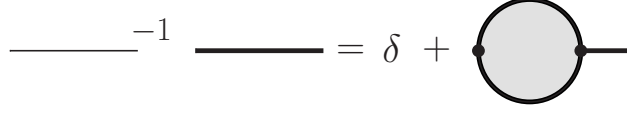


Figure 4: Diagrammatic representation of the Schwinger-Dyson equations, where the thin lines represent free (tree-level) propagators $i\Delta^{(0)}$ and solid lines the full propagators $i\Delta$. The thin line with the superscript -1 denotes the inverse of the free propagator, *i.e.* the Klein-Gordon operator. The shaded circle represents the sum of all self-energy diagrams that result from the functional derivative of the two-particle-irreducible effective action with respect to the full propagator.

such that we obtain

$$i\Delta^{(0)R}(x; x') = \frac{H^2}{4\pi^2} \left[\frac{1}{y^<(x; x')} - \frac{1}{y^T(x; x')} - i\pi\vartheta(\Delta x^2)\vartheta(\Delta x^0) \left| \frac{1}{y(x; x')} \right|^{\frac{1}{3}\frac{m^2}{H^2}} + \dots \right], \quad (37a)$$

$$i\Delta^{(0)A}(x; x') = \frac{H^2}{4\pi^2} \left[\frac{1}{y^>(x; x')} - \frac{1}{y^T(x; x')} - i\pi\vartheta(\Delta x^2)\vartheta(-\Delta x^0) \left| \frac{1}{y(x; x')} \right|^{\frac{1}{3}\frac{m^2}{H^2}} + \dots \right]. \quad (37b)$$

3.2 Schwinger-Dyson Equations

The Schwinger-Dyson equations on the CTP are

$$a^4 (\nabla_x^2 - m^2) i\Delta^{fg}(x; x') = \delta^{fg} i\delta^4(x - x') - i \int d^4w i\Pi^{fh}(x; w) i\Delta^{hg}(w; x'), \quad (38)$$

where a summation over $h = \pm$ is implied. We have introduced here the full propagator $i\Delta$, to be distinguished from the free propagator $i\Delta^{(0)}$. The self-energy Π is derived from

$$\Pi^{fg}(x; x') = i \frac{\delta\Gamma_2[\Delta]}{\delta\Delta^{gf}(x'; x)}, \quad (39)$$

where Γ_2 is the two-particle irreducible (2PI) effective action, that can be computed as $-i$ times the sum of all 2PI vacuum diagrams made up of full propagators.

The Schwinger-Dyson equations (38) can be expressed in terms of diagrams as in Figure 4. They are exact equations but in practical calculations one typically aims for approximate solutions. In this work, we perform a perturbative expansion of the self-energy to two-loop order. It should therefore be clear that the propagator $i\Delta$ that we obtain below is only an approximation (which is perturbatively improved compared to $i\Delta^{(0)}$) to the full propagator, even though we do not introduce an extra symbol for this quantity.

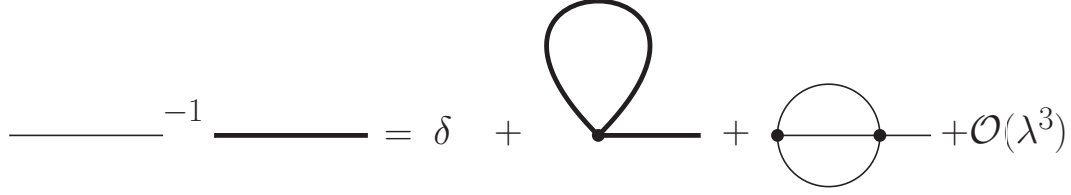


Figure 5: Diagrammatic representation of the Schwinger-Dyson equations that are perturbatively truncated at order λ^2 . Again, thin lines represent free (tree-level) propagators $i\Delta^{(0)}$ and solid lines the full propagators $i\Delta$. For our present approximation, that accounts for the leading IR effects to order λ^2 , it is sufficient to approximate the full propagator on the right-hand side of these equations by accounting for the seagull-type mass correction, *i.e.* replacing $m^2 \rightarrow m^2 + \delta m^2$ in Eq. (31).

On the CTP, we can take various linear combinations of the Schwinger-Dyson equations. A particularly useful one is

$$a^4(-\nabla_x^2 - m^2)i\Delta^{<, >}(x; x') = -i \int d^4w i\Pi^R(x; w)i\Delta^{<, >}(w, x') \quad (40a)$$

$$-i \int d^4w i\Pi^{<, >}(x; w)i\Delta^A(w, x'),$$

$$a^4(-\nabla_x^2 - m^2)i\Delta^{R,A}(x; x') = i\delta^4(x; x') - i \int d^4w i\Pi^{R,A}(x; w)i\Delta^{R,A}(w, x'), \quad (40b)$$

where Eqs. (40a) are known as the Kadanoff-Baym equations.

3.3 Organisation of the Calculation

The main goal of the calculation that is presented here is to solve the Schwinger-Dyson equations (40) (*cf.* also Figure 4) for the propagator $i\Delta$. As these are non-linear integro-differential equations, we aim for approximate perturbative solutions that capture the leading IR-effects. For that purpose, we employ the ansatz that the loop effects can be approximated by a full propagator that satisfies the free Klein-Gordon equations with a dynamical mass m_{dyn} . This amounts to replacing $m^2 \rightarrow m_{\text{dyn}}^2$ in Eqs. (28,30,31). Perturbativity is ensured by the relation $\lambda \ll m^4/H^4$. Then, there are two elementary loop contributions to the self-energy up to order λ^2 : the seagull diagram that is given in Figure 1(A) and the sunset diagram in Figure 1(D). We denote the seagull-type self energy by Π_{sg} and the sunset-type by Π_{ss} .

We now comment on the dynamical mass ansatz and the truncation of the loop expansion in more detail:

- The seagull contribution [Figure 1(A)] is manifestly local, and hence it is immediately clear that it takes the effect of a mass correction. For Euclidean de Sitter

space, it has been demonstrated that the leading IR effects in massless ϕ^4 theory, that also include an infinite number of non-local diagrams, can be effectively described by a dynamical mass term [27]. The mass square is then inversely proportional to the fluctuation of ϕ . For Lorentzian space, it is shown that local effective equations of motion can be obtained upon acting on the Schwinger-Dyson equations with the Klein-Gordon operator [15]. However, certain contributions to the effective equations of motion have been missed in that study, and we show here how to correctly calculate these for light, perturbative ϕ^4 theory. The results presented in this work therefore explicitly show the validity of the dynamical mass ansatz to two-loop order, including in particular the non-local sunset diagram [Figure 1(D)].

- In the present context, it is useful to review the IR convergence property of the loop expansion in Euclidean de Sitter space, because it is rather straightforward [27]: Within a Feynman diagram, each propagator contributes an IR-enhanced factor $\sim H^4/m^2$ and each vertex a factor of λ . Adding one vertex to a given diagram, implies two more propagators and a volume integral yielding a factor $\sim H^{-4}$. Hence, the expansion parameter can be identified to be $\lambda H^4/m^4$. For a massless field, it is found that one can replace m^2 with m_{dyn}^2 , where $m_{\text{dyn}}^2 \sim \sqrt{\lambda} H^2$, such that diagrams at all loop orders have the same degree of IR enhancement [27]. In turn, in presence of a non-vanishing tree-level mass m , the loop expansion is valid provided $\lambda \ll m^4/H^4$, in agreement with the conclusion from the stochastic approach in Section 2.
- In Lorentzian space, the seagull diagram has the same degree of IR enhancement as in Euclidean space [8, 15]. For the sunset contributions, as we discuss below, there occur four propagators in the convolution integrals on the right-hand side of the Kadanoff-Baym equation (40a) (one explicit propagator and three implicit ones within $i\Pi_{\text{ss}}$). However, it turns out that at least one of these propagators is retarded or advanced, such that it exhibits no IR-enhancement according to Eqs. (37). Superficially, it may therefore appear that the sunset contribution in Lorentzian space is less IR-enhanced (by one order in H^2/m^2) than in Euclidean space. As it is shown below, this is however not the case, because the convolution integral itself contributes an extra factor of H^2/m^2 , due to the mild power-law decay of the relevant contributions to the propagator, *cf.* Eq. (31). Up to two-loop order, the IR convergence properties in Lorentzian de Sitter space therefore turn out to effectively agree with those in Euclidean space, and one may conjecture that this extends to all orders. For this present calculation, we therefore take $\lambda \ll m^4/H^4$, such that the perturbation expansion is valid. This also implies that the number of loops within a diagram is the same as the order of λ in the perturbative expansion. The relevance of the mild decay of the IR-enhanced terms at large distances is also emphasised in [31].
- Now, calculating the leading IR corrections to the two-point functions up to order λ^2 can be done by evaluating the four diagrams that are given in Figure 1.

Instead, we make use here of the Schwinger-Dyson equations that are diagrammatically represented in Figure 4. When truncated at the perturbative order λ^2 , the Schwinger-Dyson equations take the form given in Figure 5. Note that the full (bold) propagators are approximated following the dynamical mass ansatz. As it is indicated in Figure 5, we can evaluate the sunset diagram using the free propagator, because its leading contribution is of order λ^2 , and additional corrections from using the full propagator are therefore of order λ^3 and higher. Similarly, for the full propagators that are substituted into the seagull diagram, we can use an approximation of the dynamical mass-square m_{dyn}^2 that is accurate up to order λ , as the leading seagull contribution is already of order λ . We denote the seagull diagram with the full propagator by $i\Pi_{\text{SG}}$, in order to distinguish it from the seagull diagram $i\Pi_{\text{sg}}$ with the free propagator. Since we choose the parameters such that perturbation theory is valid, we note the relation

$$i\Pi_{\text{SG}} = i\Pi_{\text{sg}} + i\Pi_{\text{ca}} + \mathcal{O}(\lambda^3). \quad (41)$$

In order to extract the leading IR effects, we only need to keep the highest powers of the enhancement factor H^2/m^2 that occur for each order in λ .

While the present perturbative calculation does not make use of the full power of the Schwinger-Dyson approach to address non-perturbative problems³, we yet find it useful because it automatically includes the Feynman-diagram contributions from Figures 1(B) and 1(C). Moreover, we anticipate that Schwinger-Dyson equations should be used as well in a future calculation for massless ϕ^4 theory in de Sitter space, that presumably requires a resummation of diagrams at all loop orders.

3.4 Solution for a Dynamical Mass Ansatz

Provided the relation (34) is satisfied, the propagator (31), which is the solution to the Klein-Gordon equation (30), can be expressed as

$$i\Delta^{(0)fg}(x; x') = \frac{H^2}{4\pi} \left[-\frac{1}{y^{fg}(x; x')} - \frac{1}{2} \log(-y^{fg}(x; x')) + \frac{3H^2}{2m^2} + \dots \right]. \quad (42)$$

This is the form of the propagator that is derived in Ref. [46] and that is used as well *e.g.* in Refs. [15, 48]. It does not exhibit the mild decay of the approximation (31) for large separations, when relation (34) is violated. We show below that the convolution integrals in the Schwinger-Dyson equations (40) can be analytically evaluated in the coincident approximation $x \approx x'$, such that as a consequence, the relation (34) is amply fulfilled and the approximation (42) can be used to infer the fluctuations of the field ϕ on horizon scales and not too far beyond.

³In fact, it is alternatively possible to calculate the corrections to the correlation function to $\mathcal{O}(\lambda^2)$ by simply summing the four diagrams in Figure 1.

Now, we consider a perturbation to the Klein-Gordon equation (30) by a small correction δm^2 to the mass-square as well as by an inhomogeneous constant term μ :

$$a^4(x)H^2 \left[-4y \left(1 + \frac{y}{4} \right) \frac{d^2}{dy^2} - 8 \left(1 + \frac{y}{2} \right) \frac{d}{dy} - \frac{m^2 + \delta m^2}{H^2} \right] i\Delta^{fg}(y(\Delta x)) \quad (43)$$

$$= fg\delta^{fg} i\delta^4(\Delta x) + a^4(x)\mu.$$

When identifying the term $\sim \delta m^2$ with the full seagull correction $i\Pi_{\text{SG}}$ and the term $\sim \mu$ with the sunset correction $i\Pi_{\text{ss}}$ [the precise form of δm^2 and μ is given by Eqs. (49,51) below], this equation corresponds to the Schwinger-Dyson equation that is truncated at order λ^2 that is discussed in Section 3.3 above and that is represented by Figure 5.

For a solution to Eq. (43) that is valid on scales where the relation (34) is satisfied, we make the ansatz

$$i\Delta(y) = -\frac{H^2}{4\pi^2 y} + A \log y + C + \dots \quad (44)$$

The normalisation is imposed by the inhomogeneous δ -function term, while A and C are coefficients to be determined. This ansatz is also an approximate solution to the free Klein-Gordon equation (30) when replacing $i\Delta^{(0)} \rightarrow i\Delta$ and $m^2 \rightarrow m_{\text{dyn}}^2 = \frac{3H^4}{8\pi^2 C}$. This is why it qualifies as a ‘dynamical mass’ ansatz, even though we find it more convenient here to approximate the full propagator using the parameter C .

Substituting the ansatz (44) into Eq. (43) and collecting the terms $\propto 1/y$ and the constant terms, we obtain for the coefficients in the ansatz (44):

$$A = -\frac{H^2}{8\pi^2} + H^2 \mathcal{O} \left(\frac{m^2 + \delta m^2}{H^2} \right), \quad (45a)$$

$$C = -\frac{3H^2}{m^2 + \delta m^2} A - \frac{\mu}{m^2 + \delta m^2} \approx \frac{3H^4}{8\pi^2(m^2 + \delta m^2)} - \frac{\mu}{m^2} \quad (45b)$$

$$\approx \frac{3H^4}{8\pi^2 m^2} - \frac{3H^4 \delta m^2}{8\pi^2 m^4} + \frac{3H^4 \delta m^4}{8\pi^2 m^6} - \frac{\mu}{m^2}.$$

The last two approximations are valid to order λ^2 , when we identify $\delta m^2 \sim i\Pi_{\text{SG}}$ and $\mu \sim i\Pi_{\text{ss}}$. (These identifications are made more explicit and quantitative in Section 3.5.) Note that we need to account for the dependence of δm^2 on C , in order to obtain a self-consistent solution for C , that is valid up to order λ^2 . In fact, we can then interpret within the last expression of Eq. (45b) the second term as the sum of the seagull and the cactus diagram in Figures 1(A) and 1(C) (provided Π_{SG} and hence δm^2 are accurate to order λ^2), the third term as the diagram in Figure 1(B) and the fourth term as the sunset diagram, Figure 1(D). This shows that indeed, the Schwinger-Dyson approach is equivalent here to a straightforward perturbative calculation. We also observe that at leading order in H^2/m^2 , a mass-square perturbation δm^2 takes the same effect as a constant inhomogeneous term $8\pi^2 \mu m^2 / (3H^4)$.

3.5 Evaluation of the Self-Energies and Solution to the Schwinger-Dyson equations

Now, we show that the leading IR contributions to the self-energy terms on the right hand side of the Kadanoff-Baym equations (40a) can indeed be effectively accounted for by the terms δm^2 and μ as in the perturbed the Klein-Gordon equation (43). We moreover evaluate these corrections and compare the result for the full propagator to the stochastic fluctuations.

The sunset contribution [Figure 1(D)] to the self-energy is given by

$$i\Pi_{\text{ss}}^{fg}(x; x') = \frac{\lambda^2}{6} a^4(x) \left(i\Delta^{(0)fg}(x; x') \right)^3 a^4(x'). \quad (46)$$

Using this expression, we evaluate the first of the convolution integrals in Eq. (40a) for $x = x'$:

$$\begin{aligned} & -i \int d^4w i\Pi_{\text{ss}}^{<, >}(x; w) i\Delta^{(0)A}(w; x) \\ &= -i \int d^4w \frac{\lambda^2}{6} a^4(x) \left(i\Delta^{(0)<, >}(x; w) \right)^3 a^4(w) i\Delta^{(0)A}(w; x) \\ &\approx -\frac{9\lambda^2 H^{14}}{2^{12}\pi^8 m^6} a^4(x) \int d^4w a^4(w) |y(x; w)|^{-\frac{4}{3}\frac{m^2}{H^2}} \pi \vartheta((w^0 - x^0)^2 - (\mathbf{w} - \mathbf{x})^2) \vartheta(w^0 - x^0) \\ &\approx -a^4(x) \frac{9\lambda^2 H^{12}}{2^{12}\pi^6 m^8}. \end{aligned} \quad (47)$$

For the first approximation, we have extracted the leading IR contributions that are enhanced by factors of H^2/m^2 . The details for the calculation of the last integral and the approximations made are presented in Appendix B. A similar calculation yields an approximation for the second convolution integral in Eq. (40a):

$$-i \int d^4w i\Pi_{\text{ss}}^R(x; w) i\Delta^{(0)<, >}(w; x) \approx -a^4(x) \frac{27\lambda^2 H^{12}}{2^{12}\pi^6 m^8}. \quad (48)$$

We note here that while only three explicitly IR-enhanced factors H^2/m^2 occur within the integrands of the convolution integrals, as the retarded and advanced propagators (37) do not contain such terms, the results of the integrals are nonetheless of fourth order in H^2/m^2 . This is because the integration yields the missing factor due to the mild decay of the propagators at large distances, which can be seen from Eq. (31). By comparing Eq. (43) with Eq. (40a), we identify

$$\mu = -\frac{9\lambda^2 H^{12}}{2^{10}\pi^6 m^8} + \mathcal{O}(\lambda^3). \quad (49)$$

Note that $-\mu$ agrees with the corresponding result $\langle \phi^2 \rangle_{\text{ss}}$ from the sunset diagram in the stochastic approach Eq. (14d).

We also quote the the leading IR results for the seagull-type self-energies and convolutions, $i\Pi_{\text{SG}}^{<, >}(x; w) = 0$ and

$$-i \int d^4 w i\Pi_{\text{SG}}^R(x; w) i\Delta^{<, >}(w; x') = a^4(x) \frac{\lambda}{2} C i\Delta(x; x') = a^4(x) \delta m^2 i\Delta(x; x'), \quad (50)$$

such that

$$\delta m^2 = \frac{\lambda}{2} C. \quad (51)$$

We note that the term $-3H^4\delta m^2/(8\pi^2 m^4)$ in Eq. (45b) can be identified with the sum of the seagull and the cactus diagram, Figures 1(A) and 1(C). Furthermore, the result agrees with the corresponding diagrammatic contributions in the stochastic approach, $\langle\phi^2\rangle_{\text{sg}} + \langle\phi^2\rangle_{\text{ca}}$, see Eqs. (14a, 14b). Similarly, the term $3H^4\delta m^4/(8\pi^2 m^6)$ can be identified with the diagram in Figure 1(C), and it agrees with the stochastic result, $\langle\phi^2\rangle_{\text{sgsg}}$, Eq. (14c). Together with the fact that $-\mu^2 = \langle\phi^2\rangle_{\text{ss}}$, this implies that to $\mathcal{O}(\lambda^2)$, the individual Feynman diagrams in the CTP formalism (after summing over CTP indices on internal vertices) evaluate to the same results as the corresponding diagrams in the stochastic approach. Of course, it would be interesting to generalize this statement beyond the two-loop order.

Now, we substitute Eqs. (49) and (51) into Eq. (45b) and solve for the coefficient C , such that we obtain

$$C = \frac{3H^4}{8\pi^2 m^2} - \frac{9\lambda H^8}{128\pi^4 m^6} + \frac{9\lambda^2 H^{12}}{256\pi^6 m^{10}} + \dots, \quad (52)$$

which is in agreement with the result (15) derived by stochastic methods. Note that we may also infer a dynamical mass through

$$m_{\text{dyn}}^2 = \frac{3H^4}{8\pi^2 C} = m^2 + \frac{3\lambda H^4}{16\pi^2 m^2} - \frac{15\lambda^2 H^8}{256\pi^4 m^6} + \mathcal{O}(\lambda^3). \quad (53)$$

4 Generalisation to D Space-Time Dimensions

We now explain how to generalise above calculation to D space-time dimensions and present the result for the field fluctuations. In D dimensional de Sitter space, the action is given by

$$S = \int d^D x \mathcal{L} \quad (54)$$

and the Klein-Gordon equation is:

$$a^D(x) H^2 \left[-4y \left(1 + \frac{y}{4} \right) \frac{d^2}{dy^2} - 2D \left(1 + \frac{y}{2} \right) \frac{d}{dy} - \frac{m^2}{H^2} \right] i\Delta^{(0)fg}(y) = fg\delta^{fg}\delta^D(\Delta x). \quad (55)$$

We can also generalise Eq. (43) to D dimensions by replacing in Eq. (55) $i\Delta^{(0)} \rightarrow i\Delta$, $m^2 \rightarrow m^2 + \delta m^2$ and adding the term $a^D(x)\mu$ to the right-hand side. For the solution, we make the ansatz

$$i\Delta(y) = -\frac{2}{(D-1)KH^2y} + A \log y + C + \dots, \quad (56)$$

where the normalisation is imposed again by the inhomogeneous δ -function term in Eq. (55), and where we define

$$K = \frac{2\pi^{\frac{D+1}{2}}}{\Gamma\left(\frac{D+1}{2}\right)H^D}. \quad (57)$$

Substituting the ansatz (56) into Eq. (55) and comparing the coefficients then yields

$$A = -\frac{1}{(D-1)KH^2} + \dots \quad (58)$$

and

$$C = -\frac{(D-1)H^2}{m^2 + \delta m^2}A - \frac{\mu}{m^2 + \delta m^2} \approx \frac{1}{K(m^2 + \delta m^2)} - \frac{\mu}{m^2 + \delta m^2}. \quad (59)$$

It remains to determine the D -dimensional expressions for δm^2 and μ . The expression (51) for δm^2 is valid in D dimensions as well. For the convolution integrals, we obtain

$$\begin{aligned} -i \int d^D w i \Pi_{ss}^{<, >}(x; w) i \Delta^{(0)A}(w; x) &= -i \int d^D w \frac{\lambda^2}{6} a^D(x) i (\Delta^{(0)<, >}(x; w))^3 a^D(w) \quad (60a) \\ &\times i \Delta^{(0)A}(w; x) = -a^D(\eta) \frac{\lambda^2}{24K^3 m^8}, \end{aligned}$$

$$-i \int d^D w i \Pi_{ss}^R(x; w) i \Delta^{(0)<, >}(w; x) = -a^D(\eta) \frac{\lambda^2}{8K^3 m^8}. \quad (60b)$$

We can hence identify

$$\mu = -\frac{\lambda^2}{6K^3 m^8}. \quad (61)$$

Substituting this result and the expression (51) for δm^2 into Eq. (59), we obtain

$$C = \frac{1}{Km^2} - \frac{\lambda}{2K^2 m^6} + \frac{2\lambda^2}{3K^3 m^{10}} + \dots. \quad (62)$$

Note that within the term of $\mathcal{O}(\lambda^2)$, four factors of K^{-1} originate from the fourth power of the propagator given in Eq. (A.6) while a factor of K results from the convolution integral. The fact that up to order λ^2 (and perhaps beyond) the dependence of the

term of order λ^n on the space-time dimension is proportional to K^{-n-1} is therefore a non-trivial result of the QFT calculation. In contrast, this feature emerges from the stochastic approach in a rather obvious manner. We can also turn the result for C around in order to obtain a dynamical mass

$$m_{\text{dyn}}^2 = \frac{1}{KC} = m^2 + \frac{\lambda}{2Km^2} - \frac{5\lambda^2}{12K^2m^6} + \dots \quad (63)$$

Note that the dynamical mass is inferred here from the field fluctuations, whereas the effective infrared mass in Ref. [31] governs the decay of the two-point function in the far infrared. The two quantities are therefore not directly comparable and do in fact differ ⁴. As for the details of the calculation, in the present work, we remain in position space and aim to obtain the intermediate and final results in a manifestly de Sitter invariant manner, whenever that is possible. A particular reference frame [through the conformal coordinates in Eq. (21)] has to be chosen however in order to perform the convolution integrals as described in Appendix B. At that stage, it appears unavoidable to give up the manifest de Sitter invariance. In contrast, in Ref. [31], a mixed representation for the two-point functions is chosen, where the spatial coordinates are expressed in momentum space, and manifest de Sitter invariance is lost throughout the calculation. Of course, the final results should preserve de Sitter invariance in both approaches.

Again, we compare the field fluctuation (62) with the corresponding result from the stochastic approach. The 4-dimensional derivation for the stochastic fluctuations from Refs. [37, 38] is generalised to D dimensions in Ref. [27], where it is shown that the appropriate probability distribution is

$$\varrho(\phi) = \mathcal{N}e^{-KV(\phi)} \quad (64)$$

Using this within Eqs. (17) and (18), we obtain that

$$\langle \phi^2 \rangle = \frac{1}{Km^2} - \frac{\lambda}{2K^2m^6} + \frac{2\lambda^2}{3K^3m^{10}} + \dots \quad (65)$$

in agreement with the result (62) derived by QFT methods.

5 Conclusions

The main results of this paper are the scalar field fluctuations (52), (62) obtained by QFT methods and the fact that these agree with the corresponding quantities derived by the stochastic method, eqs. (15) and (65). While it is well known that the two methods yield the same results for the leading IR-enhanced corrections from the local seagull diagram [Figure 1(A)], which is of relevance in the large N limit [8, 17], this is to our knowledge the first result where this agreement is generalised to a situation where a non-local self-energy diagram, *i.e.* the seagull graph from Figure 1(C), is involved. In fact,

⁴We thank F. Gautier and J. Serreau for pointing this out.

when breaking down the perturbative corrections to the individual Feynman diagrams in Figure 1, all corresponding graphs obtained by the two methods are found to agree.

While there appears to be a close relation between the expressions that one obtains when calculating the fluctuations in Euclidean de Sitter space to those from the stochastic approach [27], in that the IR enhancement results entirely from powers of Gaussian two-point functions, accounting for the IR enhancement in Lorentzian de Sitter space is a bit less straightforward: Although there is no leading order IR enhancement from the retarded and advanced propagators (37), the lack of IR power is enhanced by a space-time integral that is regulated only by the mild decay of the two-point function at large separations (31). The importance of the mild decay behaviour is also emphasised in Ref. [31]. However, when comparing the result obtained there for the dynamical mass with Eq. (63), there appears to be disagreement at $\mathcal{O}(\lambda^2)$. In the CTP formalism, there are several different terms arising from a single Feynman diagram, due to the summation over the CTP indices. As a consequence, there are two different terms on the right-hand side of the Kadanoff-Baym equations (40a). For the sunset diagram [Figure 1(D)], we need to evaluate these here separately and we find that both terms are relevant at the same order of IR enhancement, but they come with different coefficients.

To this end, the evaluation of particular Feynman diagrams therefore appears more involved in Lorentzian de Sitter space than in its Euclidean counterpart, which is not surprising, as the same can also be stated about field theory in flat space-times. Nonetheless, the agreement at the level of Feynman diagrams up to two-loop order suggests that this observation may be generalised to arbitrary loop order by extending the methods that are presented here. In Ref. [27], it is already shown how to resum the individual Euclidean Feynman diagrams to match the result from the stochastic approach. Therefore, it appears that the step that is yet missing in order to explicitly solve for the leading order IR behaviour of the vacuum state of massless ϕ^4 theory in de Sitter space may be to demonstrate that the agreement between QFT and stochastic Feynman diagrams found here extends to all loop orders. If the agreement between the QFT and the stochastic methods at the diagrammatic level up to two-loop order is not accidental, then it is remarkable that the stochastic formalism vastly simplifies the way the leading IR effects are accounted for. It would be interesting to perform a derivation of the stochastic expressions for obtaining correlation functions that relies on the CTP formalism and that does not appeal to the classical behaviour of the superhorizon modes. This may also point to a simplification of the evaluation of the Feynman diagrams in the CTP formalism in de Sitter background. Work in that direction has been performed in Ref. [42], and we will further address this point in the future [43].

While in the work [14, 27] the problem of massless ϕ^4 theory in Euclidean space is resolved at leading IR order, and since arguments in favour of an analytic continuation of correlation functions to Lorentzian space exist [39], the efforts to obtain a direct solution to the problem in Lorentzian space are justified, because they lead to insights into the dynamical behaviour of the model ⁵. This is of particular relevance for Cosmology,

⁵Beyond the leading IR order, it is reported that perturbation theory in Euclidean de Sitter space indicates correlation functions that grow in temporal separations [16]. For this reason, and also for

where the initial inflationary stage, that is often presumed to set the initial conditions for the subsequent evolution of the Universe, deviates from de Sitter space in that the Hubble rate is time-dependent, and that the approximate de Sitter epoch comes to an end and perhaps also begins at a finite time. These time dependences cannot be directly accounted for in the Euclidean approach. Nonetheless, they have important consequences for the observed perturbation spectrum and perhaps lead to a significant backreaction [3–5, 49, 50]. It therefore remains an important task to find the correct solutions for massless ϕ^4 theory and eventually also Gravitation in Lorentzian de Sitter space.

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A Long-Distance Behaviour of the Propagator

The well-known solution to Eq. (55) [and Eq. (30) for $D = 4$] is given in terms of a hypergeometric function:

$$i\Delta^{(0)}(y) = \frac{\Gamma\left(\frac{D-1}{2} + \nu\right) \Gamma\left(\frac{D-1}{2} - \nu\right)}{(4\pi)^{\frac{D}{2}} \Gamma\left(\frac{D}{2}\right)} H^{D-2} {}_2F_1\left(\frac{3}{2} + \nu, \frac{3}{2} - \nu; 2; 1 + \frac{y}{4}\right), \quad (\text{A.1})$$

where

$$\nu = \sqrt{\left(\frac{D-1}{2}\right)^2 - \frac{m^2}{H^2}}. \quad (\text{A.2})$$

In order to extract the leading behaviour for large y , we make use of the transformation formula

$$\begin{aligned} & {}_2F_1\left(\frac{D-1}{2} + \nu, \frac{D-1}{2} - \nu; \frac{D}{2}; 1 + \frac{y}{4}\right) \\ &= \frac{\left(-\frac{y}{4}\right)^{-\frac{D-1}{2} - \nu} \Gamma\left(\frac{D}{2}\right) \Gamma(-2\nu)}{\Gamma\left(\frac{D-1}{2} - \nu\right) \Gamma\left(\frac{1}{2} - \nu\right)} {}_2F_1\left(\frac{D-1}{2} + \nu, \frac{1}{2} + \nu; 1 + 2\nu; -\frac{4}{y}\right) \\ &+ \frac{\left(-\frac{y}{4}\right)^{-\frac{D-1}{2} + \nu} \Gamma\left(\frac{D}{2}\right) \Gamma(2\nu)}{\Gamma\left(\frac{D-1}{2} + \nu\right) \Gamma\left(\frac{1}{2} + \nu\right)} {}_2F_1\left(\frac{D-1}{2} - \nu, \frac{1}{2} - \nu; 1 - 2\nu; -\frac{4}{y}\right). \end{aligned} \quad (\text{A.3})$$

obtaining corrections beyond the stochastic approximation, it would be interesting to calculate sub-leading contributions in the IR expansion of the Lorentzian correlation functions, using the methods that are employed in the present work.

The desired approximation then results from the defining series expansion

$${}_2F_1(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{\Gamma(\gamma+n)} \frac{z^n}{n!}, \quad (\text{A.4})$$

such that we obtain for $D = 4$

$$\begin{aligned} i\Delta^{(0)}(y) = & H^2 \left(\frac{1}{16\pi^2} - \frac{1}{24\pi^2} \frac{m^2}{H^2} + \mathcal{O}\left(\frac{m^4}{H^2}\right) \right) \left(-\frac{y}{4}\right)^{-\frac{5}{2}+\nu} \\ & + H^2 \left(\frac{3H^2}{8\pi^2 m^2} - \frac{7}{24\pi^2} + \mathcal{O}\left(\frac{m^2}{H^2}\right) \right) \left(-\frac{y}{4}\right)^{-\frac{3}{2}+\nu} \\ & + H^2 \mathcal{O}\left(\frac{m^2}{H^2}\right) \left(-\frac{y}{4}\right)^{-\frac{7}{2}+\nu} + \dots \end{aligned} \quad (\text{A.5})$$

When keeping D general, the expressions for the higher-order terms, which are not relevant in the present context, are somewhat lengthy. We therefore only present the leading behaviour, which is

$$i\Delta^{(0)}(y) = \frac{1}{2(D-1)KH^2} \left(-\frac{y}{4}\right)^{-1-\frac{m^2}{(D-1)H^2}} + \frac{1}{m^2 K} \left(-\frac{y}{4}\right)^{-\frac{m^2}{(D-1)H^2}} + \dots, \quad (\text{A.6})$$

and where K is defined in Eq. (57).

B Convolution integral in the Kadanoff-Baym Equation

The integrations in Eqs. (47,48) can be approximately performed as follows:

$$\begin{aligned}
\mathcal{I}(x) &= a^D(x) \int d^D x' a^D(x') |y(x; x')|^{-\frac{n}{D-1} \frac{m^2}{H^2}} \vartheta((\eta - \eta')^2 - (\mathbf{x} - \mathbf{x}')^2) \vartheta(\eta' - \eta) \quad (\text{B.1}) \\
&= \frac{1}{H^D \eta^D} \int_{-\infty}^{\eta} d\eta' \frac{1}{H^D \eta'^D} \int d^{D-1} x' \vartheta((\eta - \eta')^2 - (\mathbf{x} - \mathbf{x}')^2) \\
&\quad \times \left(\frac{\eta \eta'}{[(\eta - \eta')^2 - (\mathbf{x} - \mathbf{x}')^2]} \right)^{\frac{n}{D-1} \frac{m^2}{H^2}} \\
&= \frac{1}{H^D \eta^D} \int_{-\infty}^{\eta} d\eta' \frac{2\pi^{\frac{D-1}{2}} (\eta \eta')^{\frac{n}{D-1} \frac{m^2}{H^2}}}{H^D \eta'^D \Gamma\left(\frac{D-1}{2}\right)} (\eta - \eta')^{D-1 - \frac{2n}{D-1} \frac{m^2}{H^2}} \frac{\Gamma\left(\frac{D-1}{2}\right) \Gamma\left(1 - \frac{n}{D-1} \frac{m^2}{H^2}\right)}{2\Gamma\left(\frac{D+1}{2} - \frac{n}{D-1} \frac{m^2}{H^2}\right)} \\
&\approx \frac{1}{H^{2D} \eta^D} \frac{\pi^{\frac{D-1}{2}}}{\Gamma\left(\frac{D+1}{2}\right)} \int_{-\infty}^{\eta} d\eta' (-\eta')^{-1 - \frac{n}{D-1} \frac{m^2}{H^2}} = \frac{\pi^{\frac{D-1}{2}}}{H^{2D} \eta^D \Gamma\left(\frac{D+1}{2}\right)} \left[\frac{(-\eta')^{-\frac{n}{D-1} \frac{m^2}{H^2}}}{\frac{n}{D-1} \frac{m^2}{H^2}} \right]_{\eta' \rightarrow -\infty}^{\eta' = \eta} \\
&\approx a^D(\eta) \frac{(D-1)\pi^{\frac{D-1}{2}}}{n\Gamma\left(\frac{D+1}{2}\right) H^{D-2} m^2}.
\end{aligned}$$

We also make use of this result in order to obtain Eqs. (60).

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